TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 156, May 1971

SEMIGROUPS ON FINITELY FLOORED SPACES

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Abstract. This paper is concerned with certain aspects of acyclicity in a compact connected topological semigroup, and applications to the admissibility of certain multiplications on continua. The principal result asserts that if S is a semigroup on a continuum, finitely floored in dimension 2, then $S=ESE$ implies $S=K$.

Introduction. We are concerned here with some aspects of acyclicity in a compact connected semigroup, and applications to the admissibility of a semigroup structure on "finitely floored" spaces (Definition 2). The main result is that if S is a semigroup on a continuum, finitely floored in dimension 2, then $S=ESE$ implies $S=K$. This is a generalization of the result of Cohen and Koch [1], where the same conclusion is obtained assuming that S is a floor for each nonzero $h \in H^2(S)$, cd $S \leqq 2$, and S is locally Euclidean except possibly at one point.

A preliminary result which may be of interest in itself is that if S is a semigroup with a zero satisfying $S=ESE$ on a continuum, then for each nonzero $h \in H^2(S)$ there exists a pair of idempotents e and f such that $h|Se \cup Sf \neq 0$.

The notation is that of [4]. In particular, S denotes a topological semigroup, K the minimal ideal, and E the set of idempotents. For a closed set A of S, S/A denotes the space obtained by shrinking A to a point. The cohomology used is that of Alexander-Wallace-Spanier with coefficient group arbitrary. Throughout the paper we shall use reduced groups in dimension 0. We denote by $A \backslash B$ the complement of B in A, the closure of A by A^* and the empty set by $\square$. If $\leqq$ is a relation from a space X to a space Y, and $y \in Y$, then $L(y)=\{x \in X : x \leqq y\}$; if $B \subset Y$, then $L(B)=\{x \in X : x \leqq b \text{ for some } b \in B\}$.

The author would like to express his gratitude to Professor R. J. Koch for his many useful suggestions during the preparation of this paper.

Topological preliminaries. If A is a closed subset of a space X and h is an element of $H^n(X)$, then $h|A$ denotes the image of h under the natural homomorphism $H^n(X) \xrightarrow{i^*} H^n(A)$. The following theorem is due to Wallace [5]:

THEOREM 1. *Let X and Y be compact Hausdorff spaces and $\leqq$ a closed relation from X to Y such that $L(A) \cap L(B)$ is connected for each pair of closed subsets A and*

Received by the editors June 29, 1970.

AMS 1970 *subject classifications*. Primary 22A15; Secondary 22A15.

Key words and phrases. Semigroups on continua, acyclicity in compact semigroups, admissibility of multiplication on two-manifolds.

B of Y. If $h \in H^1(X)$ has the property that $h|L(y)=0$ for each $y \in Y$, then $h|L(A)=0$ for each closed subset of Y.

DEFINITION 2. Let X be a space and $L \neq \square$ a subset of $H^n(X)$. A closed subset F of X is called a *floor* for L if (i) $h|F \neq 0$ for each $h \in L$ and (ii) for each proper closed subset A of F, there exists an element h in L such that $h|A=0$. If L is finite and $F=X$, X is said to be *finitely floored* in dimension n.

It should be observed that if X is compact and $L \subset H^n(X)$ does not contain 0, then there exists a floor F for L. In particular, if A is a closed subset of X such that $h|A \neq 0$ for each $h \in L$, then F may be chosen as a subset of A.

The following topological lemma is a modified version of that due to Cohen and Koch [1].

LEMMA 3. *Let X be a continuum such that $H^n(X) \neq 0$, and suppose X is a floor for some subset L of $H^n(X)$. If A is a proper retract of X, then $H^n(X/A) \neq 0$ and X/A is a floor for some subset L_0 of $H^n(X/A)$. Moreover, L_0 may be chosen to be finite if L is finite.*

Proof. Let φ: $X \to X/A$ denote the natural map and $p=\varphi(A) \in X/A$. Consider the commutative diagram in which the top row is exact and φ_0^* is induced by the natural map φ_0: $(X, A) \to (X/A, p)$.

$$\begin{array}{ccccc} H^n(X, A) & \longrightarrow & H^n(X) & \longrightarrow & H^n(A) \\ \varphi_0^* \uparrow & & \varphi^* \uparrow & & \\ H^n(X/A, p) & \longrightarrow & H^n(X/A) & & \end{array}$$

Note that φ_0^* is an isomorphism by the map excision theorem [6]. Let $L_1 \subset L$ be those elements h in L satisfying $h|A=0$. For each $h \in L_1$ there exists an element h_0 of $H^n(X/A)$ such that $\varphi^*(h_0)=h$. Let $L_0 \subset H^n(X/A)$ be a "section" of L_1; i.e. for each $h \in L_1$ there exists a unique element h_0 in L_0 such that $\varphi^*(h_0)=h$. Clearly such a set exists and is finite if L is finite. We shall show that X/A is a floor for L_0.

By the manner in which h_0 was chosen, it is clear that $0 \notin L_0$. Suppose that B is a proper closed subset of X/A and consider the commutative diagram

$$\begin{array}{ccc} H^n(X) & \longrightarrow & H^n(\varphi^{-1}(B)) \\ \varphi^* \uparrow & & \varphi_1^* \uparrow \\ H^n(X/A) & \longrightarrow & H^n(B) \end{array}$$

φ_1^* is induced by the restriction of φ to $\varphi^{-1}(B)$. Since X is connected, $\varphi^{-1}(B) \cup A$ is a proper closed subset of X. Hence there is an element h in L such that $h|\varphi^{-1}(B) \cup A =0$. In particular, $h|\varphi^{-1}(B)=0$ and $h|A=0$, $h \in L_1$, and so there exists an element h_0 in L_0 such that $h_0|B=0$. Our intention is to show that $h_0|B=0$. Since $h|\varphi^{-1}(B)=0$ and $h|\varphi^{-1}(B)=\varphi^*(h_0)|\varphi^{-1}(B)=\varphi_1^*(h_0|B)$, it suffices to show that φ_1^* is injective.

With this end in mind, two cases are considered:

(i) $p=\varphi(A)\in B$ and

(ii) $p=\varphi(A)\notin B$.

In case (i) φ_1 is a homeomorphism of $\varphi^{-1}(B)$ onto B and thus φ_1^* is an isomorphism. In case (ii) we consider the diagram

$$\begin{array}{ccc} H^n(B) & \xrightarrow{\varphi_1^*} & H^n(\varphi^{-1}(B)) \\ {\scriptstyle k^*}\uparrow & & \uparrow{\scriptstyle i_0^*} \\ H^n(B,p) & \xrightarrow{\varphi_2^*} & H^n(\varphi^{-1}(B),A), \end{array}$$

where φ_2^* is induced by the restriction of φ_0 to $(\varphi^{-1}(B), A)$. Now k^* and φ_2^* are isomorphisms by the map excision theorem. It remains to show that i_0^* is injective. Consider the exact sequence

$$H^{n-1}(\varphi^{-1}(B)) \xrightarrow{m^*} H^{n-1}(A) \longrightarrow H^n(\varphi^{-1}(B), A) \xrightarrow{i_0^*} H^n(\varphi^{-1}(B)).$$

Since A is a retract of X, m^* is onto. Therefore i_0^* is injective. This completes the proof.

Principal results. The following theorem is of use in the sequel and is of independent interest in itself.

THEOREM 4. *Let S be a continuum with a zero satisfying $S=ESE$. If h is a nonzero element of $H^2(S)$, then there exists a pair of idempotents e and f of S such that $h|Se\cup Sf\neq 0$.*

Proof. Let F be a subset of E, minimal relative to (i) being closed and (ii) satisfying $h|SF\neq 0$. (Here we are using the hypothesis that $S=SE$.) Since S has a zero, $H^2(Se)=0$, so F does not consist of a single element. Thus F may be expressed as the union of two proper closed subsets A and B of F. Consider the Mayer-Vietoris sequence

$$\longrightarrow H^1(SA\cap SB) \xrightarrow{\Delta} H^2(SF) \xrightarrow{J^*} H^2(SA)\times H^2(SB) \longrightarrow.$$

Because of the minimal conditions of F, $h|SA=0$ and $h|SB=0$; thus $J^*(h|SF)=0$. Hence there exists an element h_0 of $H^1(SA\cap SB)$ such that $\Delta(h_0)=h|SF$. Define a relation $\leqq$ from $SA\cap SB$ to $A\times B$ as follows: For $x\in SA\cap SB$ and $(e,f)\in A\times B$, let $x\leqq(e,f)$ if and only if $x\in Se\cap Sf$. It is easily verified that $\leqq$ is a closed relation from $SA\cap SB$ to $A\times B$. For each subset C of $A\times B$, $L(C)=\bigcup\{Se\cap Sf : (e,f)\in C\}$; therefore $L(C)$ is a left ideal of S. Because S has a zero and $S=ES$, it is easily verified that the left ideal $L(M)\cap L(N)$ is connected for each pair of closed subsets M and N of $A\times B$. Now $h_0|L(A\times B)=h_0|SA\cap SB=h_0\neq 0$, so by Theorem 1 it follows that there exists a pair $(e,f) \in A \times B$ such that $h_0 | L((e,f)) \neq 0$. That is to say, $h_0 | Se \cap Sf \neq 0$. Consider the commutative diagram

$$\begin{array}{ccccc} & & H^1(SA \cap SB) & \xrightarrow{\Delta} & H^2(SF) \\ & & \downarrow & & \downarrow \\ H^1(Se) \times H^1(Sf) & \longrightarrow & H^1(Se \cap Sf) & \xrightarrow{\Delta_0} & H^2(Se \cup Sf). \end{array}$$

Since S has a zero, Δ_0 is injective. Therefore $h | Se \cup Sf = \Delta_0(h_0 | Se \cap Sf) \neq 0$ and the proof is complete.

A point p in a space X is said to be *peripheral* if there exist small neighborhoods V containing p such that $H^n(V^*, V^* \backslash V) = 0$ for all nonnegative n. The following lemma follows from Hofmann and Mostert [3, p. 168] and the definition.

LEMMA 5. *Let S be a continuum and e an idempotent of $S \backslash K$. If $e \in (Se)^0$, then e is peripheral in S.*

LEMMA 6. *Suppose that X is a continuum and X is a floor for some subset L of $H^n(X)$; then no point of X is peripheral.*

Proof. For each neighborhood V of X the natural homomorphism of $H^n(X) \xrightarrow{i^*} H^n(X \backslash V)$ is not injective. Indeed there exists a nonzero element h of L such that $h | X \backslash V = 0$. The conclusion follows from the fact that $H^n(X, X \backslash V)$ is isomorphic to $H^n(V^*, V^* \backslash V)$ under the natural homomorphism.

THEOREM 7. *Let S be a continuum satisfying $S = ESE$. If S is a floor for some finite subset L of $H^2(S)$, then $S = K$.*

Proof. Since S is a floor for L, $H^2(S) \neq 0$. If $S \neq K$, then K is a proper retract of S [7]. Thus the hypothesis of Lemma 3 is satisfied, and so S/K, the Rees quotient modulo K, satisfies the hypothesis of the theorem. We may now assume that S has a zero.

By Theorem 4 for each $h \in L$ there exists a pair of idempotents e_h and f_h in S such that $h | Se_h \cup Sf_h \neq 0$. Let $A = \bigcup \{Se_h \cup Sf_h : h \in L\}$ and observe that A is closed because L is finite. Clearly $h | A \neq 0$ for each $h \in L$ and hence $A = S$. We conclude that there exists a finite subset $\{e_i\}_{i=1}^n$ of E such that $S = \bigcup_{i=1}^n Se_i$, and also that $e_i \notin Se_j$ for $i \neq j$. Then $e_1 \in (Se_1)^0$, so by Lemma 5, e_1 is peripheral in S. This establishes a contradiction to Lemma 6, and the proof is complete.

The class of compact connected 2-manifolds without boundary are covered by the preceding theorem, and other examples are readily constructed. Also, it should be noted that there is an example of a semigroup S with a zero satisfying $S = ESE$ and $H^2(S) \neq 0$ [2]. The underlying space of S is a 2-sphere with four closed intervals issuing from a common point z on the 2-sphere. The point z is a zero for S and the other idempotents for S are the free endpoints of the four arcs. In view of Theorems 4 and 7 this example is in a sense a prototype of any such example.

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